WHO GOES FIRST?

By Eric Harshbarger

It's not often that one can claim that he has helped advance the technology of board gaming. I mean, once the board had been invented, dice created, and cards used, what more is there? Counting pegs? Poker chips? The scoring track that encircles many boards these days—what was the first game to use that mechanic? The rules of board games these days may vary greatly, but rarely does the underlying technology get an upgrade.

So, it is with much enthusiasm that I report the following story about how I had a hand in co-inventing something that can help the players of just about any board game out there. Some cynics may doubt the utility of this invention, and others may question whether or not it constitutes an actual "advance of technology," but none can refute the validity of it. And for anybody who has ever struggled to decide who goes first at the start of a game, maybe this item will provide a bit of relief. Therefore, without further ado, let me relate to you the history of this marvelous invention: the set of "Go First Dice."

During the summer of 2010 I found myself having dinner with a friend in Indianapolis, Indiana. The friend was James Ernest (whom many may recognize as the founder of the popular company Cheapass Games). We were both attending the annual Gen Con gaming convention. I was helping to administer the Puzzle Hunt that year, and James was doing many things—among them promoting the board game Lords of Vegas, which he co-created.

Our talk, unsurprisingly, drifted to game design, simplicity of rules, and so forth. Playing a bit of devil's advocate, I asked James why, when we'd played Lords of Vegas earlier that day, we had rolled two dice to determine the starting player. Wouldn't it have been simpler if we'd only rolled one die (no adding required)? His response: Rolling two dice lessened the chance of ties among players. If we had only rolled one die apiece, then it would have been more likely that two or more of us would have rolled the same number, necessitating more rolling. Using two dice did not eliminate the chance of a tie, but it did lower it. Of course, we could have rolled a lot more than two dice at a time (say, ten), and the chance of tying would have been further reduced, but at some point adding up all those dice really would have become unwieldy, and rolling just a pair seemed, to James, to be a decent compromise.

This was actually a much more in-depth response than I'd expected. Certainly James seemed to have put more thought into the answer than I had put into the question when I'd half-jokingly asked it. It was an almost trivial thing to ponder, really. I mean, how much trouble is it to just re-roll dice if needed? Or, for that matter, there are plenty of other ways to determine who goes first that won't ever result in a tie (math tricks, drawing cards, etc.). But, trivial or not, the topic had come up, and as it turns out, we were just getting started.

A few minutes later James posed the following challenge to me.

"Eric, you have a background in mathematics. Here's something to think about. Can you devise a set of eight 6-sided dice used to determine who goes first in a game such that they have the following properties: First, no ties are ever rolled and, second, regardless of how many players are rolling against one another—any number of people between 2 and 8—each player always has a perfectly fair chance of rolling the high number?"

James was correct—I do have a background in mathematics (a Bachelor's and Master's degree in the subject). It was that background that helped me realize that whether or not such a set of dice was practical for game playing, mathematically it was quite an interesting question. It was also not a question that could be answered easily.

The first proposed property of the challenge was the easier one to solve. To guarantee that no ties would ever be rolled, one simply needs to not repeat any numbers on the faces of the dice. If eight 6-sided are desired, then simply enumerate them, somehow, with the numbers 1 through 48. This would ensure no two dice ever tied. (Actually, you can ensure such a property by just making sure that no number is repeated *between* dice; a particular die could repeat a number on itself, but we won't worry ourselves with that technicality because it doesn't really gain us anything.)

It was the second proposed property that was the tricky one. Each die was supposed to be fair; that is, have the same chance to roll the highest number as any of the other dice. Furthermore, not only was each die supposed to be fair when rolled against the other seven (James had requested a set of eight dice), but *any subset of the dice* should also retain this property. Meaning, for example, if five of the dice were rolled against one another (when five players are about to start a game), each would have a 1 in 5 chance of rolling high. And this was supposed to hold true for *any* five dice from the eight, regardless of which dice were chosen by which of the five players. And for any set of four dice each should have a ¼ chance of winning, any three dice a ⅓ chance, and so on. Any arbitrary subset of the dice should give each roller a perfectly fair chance of rolling the high number.

This is far from trivial to satisfy.

My immediate suggestion was to number the dice in a "serpentine" fashion. This means, given the eight dice, distribute the numbers 1 through 8 across them, then reverse your direction and put numbers 9 through 16 on them; reverse again for 17 through 24, so on (see the sidebar on page 88 for a full example). Such a configuration would help balance out the numbers. The die with the 48 on it would also have the 1 on it. The 48 would always win if rolled faceup, so this probably had to be "balanced" with the 1 on the same die because the 1 would always lose. The goal was to not have any die gain a better chance over the others, and maybe this technique would attain this.

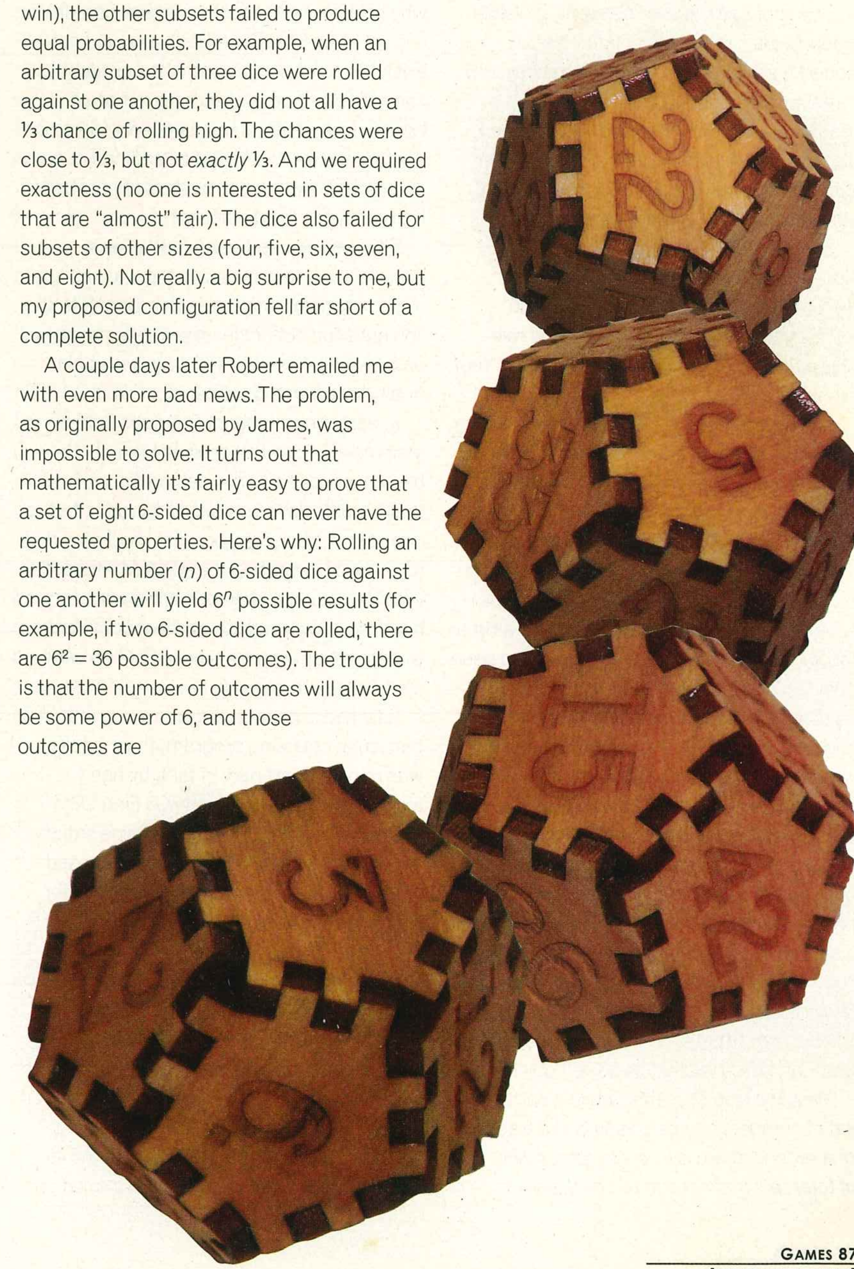
I was not up to crunching all of the calculations by hand while eating a plate full of pasta, so I could not be sure that this was an answer to the problem. In fact, I was doubtful it was a valid solution (surely it was not that simple?), but it was a decent enough suggestion to allow us to move on to other pressing topics (after all, we were at Gen Con—there were other games, costumed attendees, and late night poker tournaments to discuss).

The next week, however, upon returning home I approached the problem again. It was at this time I enlisted the help of another friend of mine, Dr. Robert Ford. Robert and I had grown up together, and he too had formally studied mathematics in college (eventually obtaining a Ph.D. in the subject from Auburn University and currently teaching mathematics at Dalton State College in Georgia). My strengths in mathematics had always leaned towards geometry, and I knew Robert had a much better feel for topics that flirted with probability (which this "Go First Dice" problem obviously did). To make sure I wasn't making bad calculations, or attempting something that could easily be proven to be impossible, I emailed Robert and related the proposed challenge. I also sent him my suggested serpentine solution.

It did not take long for Robert to shoot down my answer. He took the time to do some calculations and quickly determined that, while my proposed answer would work when any two dice were rolled against one another (each player having a ½ chance to win), the other subsets failed to produce equal probabilities. For example, when an arbitrary subset of three dice were rolled against one another, they did not all have a ⅓ chance of rolling high. The chances were close to ⅓, but not *exactly* ⅓. And we required exactness (no one is interested in sets of dice that are "almost" fair). The dice also failed for subsets of other sizes (four, five, six, seven, and eight). Not really a big surprise to me, but my proposed configuration fell far short of a complete solution.

A couple days later Robert emailed me with even more bad news. The problem, as originally proposed by James, was impossible to solve. It turns out that mathematically it's fairly easy to prove that a set of eight 6-sided dice can never have the requested properties. Here's why: Rolling an arbitrary number (n) of 6-sided dice against one another will yield 6^n possible results (for example, if two 6-sided dice are rolled, there are $6^2 = 36$ possible outcomes). The trouble is that the number of outcomes will always be some power of 6, and those outcomes are supposed to be *evenly* split up between the number of players rolling the dice. Well, in the case of five players, you can never do this. If five 6-sided dice are rolled, $6^5 = 7{,}776$ and that number is not evenly divisible by 5. It's impossible for each of the five dice to be the winner in a number of outcomes equal to all of the others. The same argument disqualifies subsets of 7 dice as well. And since certain subsets of the proposed eight 6-sided dice can *never* work, then the originally proposed challenge is impossible.

Ford offered a couple of consolation suggestions, however. First, maybe such a set of 6-sided dice would be possible if we limited ourselves to just four players maximum. Mathematically it is clear that 2 or 3 dice might have their outcomes



equally divided (since 2 and 3 each divide evenly into the number of sides on the dice: 6), and while 4 does not evenly divide into 6, it *does* evenly divide into $6^4 = 1{,}296$ (I'll try to avoid getting too mathematically technical here—the reader will just have to trust my reasoning in some places). This doesn't guarantee that such a set of four 6-sided dice can be made, but it does show that such a set cannot be proven impossible by the same reasoning that eliminates a 5-player set.

Furthermore, there are many shapes of dice besides 6-sided. If four 6-sided proved to be unaccommodating, maybe 12-sided dice would work. The regular dodecahedron has been used as a 12-sided die by players of Dungeons & Dragons since the 1970s. (Note that 8-sided dice would *not* work since no power of 8 would ever be evenly divisible among 3 players). I then pointed out to Robert that there also exists a geometrically pleasing polyhedron (the rhombic triacontahedron) that is used as a 30-sided die, and 30 is divisible by 5 and 6; so maybe such a shape could eventually yield a set of dice usable by up to six players!

But we decided to start modestly: We would tackle the four 6-sided version of the problem. (I'll abbreviate this as "4d6" following the old notation of tabletop role-playing games meaning "four die six" or "four six-sided dice").

A couple of weeks later Robert was visiting town, and when we couldn't rally together our usual group of friends to play board games, we ended up sitting around the whole day discussing our Go First Dice progress.

This is where my background in computer programming paid off (during the late '90s I spent several years working as a programmer for Sun Microsystems, Inc.). Robert had been doing all of his calculations *by hand* (well, he used a pocket calculator sometimes, I suppose). I, however, had spent some time writing computer applications that would help us in our endeavor. For example, I had written a program that could quickly check whether or not a specific configuration of numbers on a set of dice satisfied the problem as stated (that is, it would tell me if it was, in fact, a true Go First Dice set). It could "crunch the numbers" of a particular set in a few milliseconds.

"So, then, what's the trouble?" one might ask. Can't this problem be solved quickly?

Well, the trouble is that, while a particular set of numbers can be checked in a fraction of a second, there is *a very large number of total sets that need to be checked.* For example, the 4d6 case requires the placement of twenty-four numbers (1 through 24) on the faces of the dice. Considered most generally, there are 620,448,401,733,239,439,360,000 ways this can be done. Now, many restrictions can be placed on the task; some mathematical symmetries can narrow the search space down, and other requirements on the distribution of the numbers can further speed things up. But, in the end, the number of configurations that need to be checked more than overwhelms the advantage of "a few milliseconds" per check.

That particular afternoon I wrote another program that did, in fact, iterate through all of those possible 4d6 configurations. (I'll spare the reader a lengthy discussion about optimizing computer code and algorithms.) By the end of the day Robert and I learned why he'd not been able to construct a valid 4d6 set of Go First Dice: Such a set did not exist. My computer program had exhaustively checked every possible configuration of twenty-four different numbers spread across the faces of four 6-sided dice and had come up empty. There was no such configuration that defined a "Go First" set.

This was disheartening (but at least Robert now knew he could stop wasting any more time searching for such a set). With the 4d6 question out of the way, we set our sights on dice with 12 sides. And soon we would finally have some good news.

In early September of that year Robert sent me an email titled, "A Solution?" He had concocted a particular configuration that his calculations seemed to verify, but one of the checking programs I had provided to him was indicating the solution was not valid. He was still doing all of his work by hand, so he was not 100 percent confident that the error was not his own, but he wanted me to check things out.

It turned out that there was a bug in that particular checking program; the mistake was mine. Robert had, in fact, by hand, found a 4d12 solution to the Go First Dice problem. Not only did the set of dice satisfy the challenge James Ernest had proposed a month earlier, but it also had some other nice qualities to it:

• The four 12-sided dice used the numbers 1 through 48 on the faces, and Robert's solution exhibited some aesthetically pleasing symmetries. Foremost, the numbers were arranged such that each pair of opposite sides on the dice summed up to the same number: 49 (i.e., the 1 and the 48 were both on the same die—and placed on opposing faces— as were the 2 and 47, 3 and 46, and so on).

• More interestingly, however, was that we eventually realized that this configuration of numbers satisfied even more stringent conditions. Not only could they be used to determine who went first in a game (designated by the highest roll), but they could also be used to fairly determine who goes second (next highest roll), third, and fourth (depending, of course, on how many dice are actually rolled). This stronger condition held for *any* subset of the dice.

Neither of the above properties are necessary to satisfy the "Go First" question as originally posed (the solution is not unique and it is certainly possible to create sets that don't exhibit the nice symmetry or which cannot fairly determine ranking beyond first place), but we were certainly glad that our answer was "stronger" than intended.

I quickly made a few large sets of 4d12 Go First Dice out of wood for the three of us (as pictured on page 87), and soon ordered a quantity of typically sized, blank 12-sided dice. With access to a laser cutter, I was able to engrave my own custom dice, and within a week I added a Go First set to my small collection.

Since then I have manufactured many more such sets of dice and made them available on my website. The dice are white and each of the four in a set is inked with a different color. I consistently color the numbers so that the 1-Die has black ink, the 2-Die red, 3-Die green, and the 4-Die blue.

It was quite exciting to finally have a set that worked. The problem, of course, had become almost completely academic at this point. As said before, it's not as though there are not already plenty of ways to go deciding who goes first in a game, and even if you do use regular dice and have to re-roll, this is not a huge inconvenience. But mathematically the problem had been quite interesting, and for a couple of math nerds, it kept us busy. Plus, now we had a novelty item that we could show off to other gamers.

With all of that said, what then, exactly, is the configuration of numbers that we used for the Go First Dice? I have mentioned already that the 1, 2, 3, and 4 are all on separate dice, and that all opposing faces sum up to 49, but that still leaves a lot of blank faces to fill in. So, without further ado, let me present all of the numbers on all of the faces:

Die 1: 1, 8, 11, 14, 19, 22, 27, 30, 35, 38, 41, 48
Die 2: 2, 7, 10, 15, 18, 23, 26, 31, 34, 39, 42, 47
Die 3: 3, 6, 12, 13, 17, 24, 25, 32, 36, 37, 43, 46
Die 4: 4, 5, 9, 16, 20, 21, 28, 29, 33, 40, 44, 45

The numbering above is not unique; there are other configurations that will also satisfy all of the attributes I have described, but compared to the number of configurations that *do not* satisfy the "Go First" properties, such valid solutions are quite rare.

But what about more than four players?

This is still an unsolved question. Certainly many board games can accommodate five or more players, so it would be nice to develop a set of Go First Dice that is fair for more than just four people. We have not forgotten about this challenge, but for more than a year now we have not been able to answer the problem one way or the other.

We know that a set accommodating five or six players would have to use dice with at least 30 sides; but we're not sure if that is enough (maybe it fails to work in the same way that four 6-sided dice failed to yield a solution). Geometrically speaking, there are a few more sizes of polyhedra available for use. For example, a pentakis dodecahedron is a 60-sided shape that could be used as a die (though I've never seen any such shaped dice mass produced).

The big problem, of course, is that the number of configurations to check grows astronomically as you increase not only the number of dice used, but especially the number of sides on those dice (you thought that 24-digit number above in the 4d6 case was large? It's nothing when you start dealing with 150 numbers spread across five 30-sided dice). With the computer resources available to us right now, it is impossible to exhaustively check every possible 5d30 configuration. And even when making vast assumptions about how the numbers might be arranged (and in the process possibly overlooking certain solutions), we still have not been able to find a larger set of dice that work.

Maybe there are readers out there who will have some insights of their own on this problem? Maybe access to faster computers? Or stronger programming and mathematical abilities? Possibly someone else can find a set of Go First Dice for five, or even six players? If so, I would love to hear about it.

Until then, readers are encouraged to visit my website at www.ericharshbarger.org/dice/ and read more about these (and other dice) I create. Included in the section about the Go First Dice is a lengthy document exhibiting all of the possible rolls of the dice (and all subsets), proving that each is perfectly fair against each of the others.

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Serpentine Numbering

This type of numbering involves distributing sequential numbers across a set of dice, reversing direction, and weaving back and forth until all of the numbers are used. For example, to place the first 48 natural numbers onto the faces of eight 6-sided dice, one would follow this pattern:

D1	D2	D3	D4	D5	D6	D7	D8
1	2	3	4	5	6	7	8
16	15	14	13	12	11	10	9
17	18	19	20	21	22	23	24
32	31	30	29	28	27	26	25
33	34	35	36	37	38	39	40
48	47	46	45	44	43	42	41

So, for example, the third die (D3) would have the following six numbers on its faces: 3, 14, 19, 30, 35, and 46. The result helps distribute high numbers and low numbers evenly across all of the dice.

While such a configuration provides each die with a ½ chance of winning when any two dice are rolled against one another, it does not provide complete "Go First" fairness.